TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 217, 1976

A LOCAL SPECTRAL THEORY FOR OPERATORS. V: SPECTRAL SUBSPACES FOR HYPONORMAL OPERATORS[1]

BY

JOSEPH G. STAMPFLI

ABSTRACT. In the first part of the paper we show that the local resolvent of a hyponormal operator satisfies a rather stringent growth condition. This result enables one to show that under a mild restriction, hyponormal operators satisfy Dunford's C condition. This in turn leads to a number of corollaries concerning invariant subspaces. In the second part we consider the local spectrum of a subnormal operator. The third section is concerned with the study of quasi-triangular hyponormal operators.

Introduction. Let H be a Hilbert space and let $x \neq 0$ be a fixed element of H. Let $T \in L(H)$, the algebra of bounded linear operators on H. Then $f(\lambda) = (T - \lambda)^{-1}x$ is a vector valued analytic function for $\lambda \in \rho(T)$ the resolvent set of T. In many cases the function f can be extended to be analytic on an open set properly containing $\rho(x)$. We will call $\tilde{f}$ an analytic extension of f if such is the case *and* $(T - \lambda)\tilde{f}(\lambda) = x$ for λ in the domain of $\tilde{f}$. We now encounter the possibility that there may be many extensions of f and they may not be single valued, i.e. they may not agree on their common domain. However if *all* extensions do agree on their common domain for each $x \in H$ we say that T has the *single valued extension* property. Any operator $T \in L(H)$ for which the point spectrum of T is empty has the single valued extension property. Moreover all hyponormal, subnormal, and normal operators have the single valued extension property. For such an operator we designate the maximal single valued extension of $(T - \lambda)^{-1}x$ by $\tilde{x}(\lambda)$. Thus $\tilde{x}(\cdot)$ is an analytic vector valued function with the property that $(T - \lambda)\tilde{x}(\lambda) = x$ for all λ in the domain of $\tilde{x}$.

DEFINITION. *Let* $x \in H, x \neq 0$ and $T \in L(H)$. Let $\tilde{x}$ be as above. Then $\rho_T(x)$, the local resolvent set of x, is equal to the domain of $\tilde{x}$. The local spectrum of x, denoted by $\sigma_T(x)$ is equal to the complement of $\rho_T(x)$. Thus $\sigma_T(x)$

Received by the editors March 31, 1975.

AMS (MOS) subject classifications (1970). Primary 47A10, 47B20; Secondary 47A15;

Key words and phrases. Hyponormal operator, subnormal operator, local spectrum, local resolvent, quasi-triangular operator.

(1) Supported in part by a grant from the National Science Foundation, NSF GP 29006. The work was further supported by Fairchild Scholar program at the California Institute of Technology.

is a nonempty compact set in the plane. All but the nonempty property of $\sigma_T(x)$ are obvious and that follows from a standard argument. For convenience, let $\sigma_T(0) = \emptyset$.

Further discussion of the local spectrum, addition properties, examples of operators without the single valued extension property and the like can be found in [2], [10], and [17].

REMARK. We have found it convenient to assume $\sigma(T) = \sigma_C(T)$, the continuous spectrum of T at many places in §I where $\sigma_R(T) = \emptyset$ would have sufficed. Thus any hyponormal operator T can be written as $T_1 \oplus T_2$ where T_1 is normal and T_2 has no point spectrum. Since all the results of §I are obviously valid for normal operators the reader may supply this extra argument if he wishes. We will use the hypothesis $\sigma(T) = \sigma_C(T)$ freely for its simplifying effect.

§I.

DEFINITION: An operator $T \in L(H)$ is said to satisfy condition C if for every closed set $F \subset \mathbf{C}$, the linear manifold $\{x \in H: \sigma_T(x) \subset F\}$ is closed.

Our immediate goal is to show that $|\tilde{x}(\lambda)| \leqslant 1/\text{dist}[\lambda, \sigma_T(x)]$ for λ in domain $\tilde{x}$ when T is hyponormal and $\sigma(T) = \sigma_C(T)$.

We begin by quoting a result from [17].

LEMMA 1. *Let $T \in L(H)$ and let $x \neq 0$ be a fixed element of H. If $\lambda_0 \in$ domain $\tilde{x}(\cdot)$ and $\lambda_0 \in \sigma_C(T)$ then $x \in$ domain$(T - \lambda_0)^{-n}$ for $n = 1, 2, \ldots$. If T is hyponormal and $x \in$ domain$(T - \lambda_0)^{-1}$ then $x \in$ domain$(T^* - \overline{\lambda}_0)^{-1}$ and $\|(T^* - \overline{\lambda}_0)^{-1}x\| \leqslant \|(T - \lambda_0)^{-1}x\|$.*

The proof of the next lemma is left to the reader.

LEMMA 2. *Let a_n be sequence of positive (nonzero) numbers which satisfy the relation $a_1^2 \leqslant a_2$ and $a_n^2 \leqslant a_{n-1}a_{n+1}$ for $n = 2, 3, \ldots$. Then $a_1^n \leqslant a_n$ for $n = 1, 2, \ldots$.*

LEMMA 3. *Let $T \in L(H)$ be hyponormal. Let $\lambda_0 \in \sigma_C(T)$. If $\lambda_0 \in$ domain $\tilde{x}(\cdot)$ for $x \in H$ with $\|x\| = 1$, then*

$$\|(T - \lambda_0)^{-1}x\|^n \leqslant \|(T - \lambda_0)^{-n}x\|.$$

PROOF. Assume without loss of generality that $\lambda_0 = 0$. Then

$$\|T^{-1}x\|^2 = (T^{-1}x, T^{-1}x) = (T^{*-1}T^{-1}x, x) \leqslant \|T^{*-1}T^{-1}x\| \leqslant \|T^{-2}x\|.$$

(Most of the steps are justified by appeal to Lemma 1.) Similarly for fixed n we see that

$$\begin{aligned}\|T^{-n}x\|^2 &= (T^{-n}x, T^{-n}x) = (T^{*-1}T^{-n}x, T^{-(n-1)}x) \\ &\leqslant \|T^{*-1}T^{-n}x\| \, \|T^{-(n-1)}x\| \leqslant \|T^{-(n+1)}x\| \, \|T^{-(n-1)}x\|.\end{aligned}$$

If we set $a_n = \|T^{-n}x\|$ we may invoke Lemma 2 to conclude that $\|T^{-1}x\|^k \leqslant \|T^{-k}x\|$ for $k = 1, 2, \ldots$.

LEMMA 4. *Let $T \in L(H)$ with $\sigma(T) = \sigma_C(T)$. For $x \in H$ let Ω be an open set containing $\sigma_T(x)$. Let $\Gamma \subset \Omega \backslash \sigma_T(x)$ be the disjoint union of a finite number of rectifiable Jordan curves with the property that*

$$\operatorname{ind}_\Gamma(\lambda) = \begin{cases} 1 & \text{for } \lambda \in \sigma_T(x), \\ 0 & \text{for } \lambda \notin \Omega. \end{cases}$$

Then

$$(T - \lambda_0)^{-n}x = -\frac{1}{2\pi i}\int_\Gamma (\lambda - \lambda_0)^{-n}\tilde{x}(\lambda)d\lambda \quad \text{for } \lambda_0 \notin \Omega.$$

(*The minus sign occurs above because $\tilde{x}(\lambda) = (T - \lambda)^{-1}x$ instead of $(\lambda - T)^{-1}x$.*)

PROOF. The proof can be adapted from Rudin [16, Lemma 10.24]. However for completeness we will include a variation of that proof. Set

$$\beta = -\frac{1}{2\pi i}\int_\Gamma (\lambda - \lambda_0)^{-n}\tilde{x}(\lambda)d\lambda.$$

Then

$$\begin{aligned}(T - \lambda_0)^n\beta &= -\frac{1}{2\pi i}\int_\Gamma [(T - \lambda) - (\lambda - \lambda_0)]^n(\lambda - \lambda_0)^{-n}\tilde{x}(\lambda)d\lambda \\ &= -\sum_{j=0}^{n}\frac{1}{2\pi i}\int_\Gamma \binom{n}{j}(\lambda - \lambda_0)^{-j}(T - \lambda)^j\tilde{x}(\lambda)d\lambda.\end{aligned}$$

For $j > 0$, $\int_\Gamma(\lambda - \lambda_0)^{-j}(T - \lambda)^j\tilde{x}(\lambda)d\lambda = 0$ by the Cauchy theorem since $\operatorname{ind}_\Gamma(\lambda_0) = 0$.

For $j = 0$ we see that

$$-\frac{1}{2\pi i}\int_\Gamma \tilde{x}(\lambda)d\lambda = -\frac{1}{2\pi i}\int_{|\lambda|=R}\tilde{x}(\lambda)d\lambda$$

for large R by the Cauchy theorem since $\tilde{x}(\cdot)$ is analytic off $\sigma_T(x)$ and $\operatorname{ind}_\Gamma(\lambda) = \operatorname{ind}_{|z|=R}(\lambda)$ for all $\lambda \in \sigma_T(x)$. But

$$-\frac{1}{2\pi i}\int_{|\lambda|=R}\tilde{x}(\lambda)d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=R}(\lambda - T)^{-1}xd\lambda = x$$

by analytic continuation of $\tilde{x}(\cdot)$ and the power series for $(\lambda - T)^{-1}x$. Thus $(T - \lambda_0)^n\beta = x$. But then $(T - \lambda_0)^n[\beta - (T - \lambda_0)^{-n}x] = 0$ and since $\lambda_0 \in \sigma_C(T)$ we conclude that $\beta = (T - \lambda_0)^{-n}x$.

We are now ready to estimate the local resolvent of a hyponormal operator.

THEOREM 1. *Let $T \in L(H)$ be hyponormal with $\sigma(T) = \sigma_C(T)$. Let $x \in H$*

when $\|x\| = 1$. *Then*

$$\|\widetilde{x}(\lambda)\| \leqslant \frac{1}{\operatorname{dist}[\lambda, \sigma_T(x)]} \quad \textit{for } \lambda \in \rho_T(x).$$

PROOF. Fix $\lambda_0 \in \rho_T(x)$. Choose an open set $U \supset \sigma_T(x)$ such that $\operatorname{dist}[\sigma_T(x), U'] < \epsilon$ for a preassigned $\epsilon > 0$. Choose $\Gamma \subset U \backslash \sigma_T(x)$ as in Lemma 4. Then

$$\|\widetilde{x}(\lambda_0)\| = \|(T - \lambda_0)^{-1}x\| \leqslant \|(T - \lambda_0)^{-n}x\|^{1/n}$$

$$= \left\| \frac{1}{2\pi i} \int_\Gamma (\lambda - \lambda_0)^{-n} \widetilde{x}(\lambda) d\lambda \right\|^{1/n}$$

$$\leqslant \left(\frac{1}{2\pi}\right)^{1/n} l(\Gamma)^{1/n} \frac{M^{1/n}}{\operatorname{dist}[\lambda_0, \Gamma]}$$

where $l(\Gamma)$ is the length of Γ and $M = \sup\{\|\widetilde{x}(\lambda)\| : \lambda \in \Gamma\}$. Letting $n \longrightarrow \infty$ we see that

$$\|\widetilde{x}(\lambda_0)\| \leqslant \frac{1}{\operatorname{dist}[\lambda_0, \Gamma]} \leqslant \frac{1}{\operatorname{dist}[\lambda_0, \sigma_T(x)] - \epsilon}$$

(at least for ϵ small). Since $\epsilon > 0$ is arbitrary the desired inequality is proved.

COROLLARY. *Let* $T \in L(H)$ *satisfy the following conditions*:

(1) $\sigma(T) = \sigma_C(T)$, *and*

(2) $\|(T - \lambda)^{-1}x\|^n \leqslant \|(T - \lambda)^{-n}x\|$ *for* $x \in \operatorname{domain}(T - \lambda)^{-n}$ *and* $\|x\| = 1$ *and all* $\lambda \in \mathbf{C}$.

Then $\|\widetilde{x}(\lambda)\| \leqslant 1/\operatorname{dist}[\lambda, \sigma_T(x)]$ *for all* $x \in \sigma_T(x)$.

THEOREM 2. *Let* $T \in L(H)$ *be hyponormal with* $\sigma(T) = \sigma_C(T)$. *Then* T *satisfies condition* C, *i.e. for closed* $F \subset \mathbf{C}$, *the manifold*, $M = \{x \in H : \sigma_T(x) \subset F\}$ *is closed.*

PROOF. Let $x_n \in M$ where x_n converges strongly to x. We can thus assume without loss of generality that $\|x_n\| = 1$ for $n = 1, 2, \ldots$. Let U be an open set in F' (the prime denotes complement) where $\operatorname{dist}[U, F] = \delta > 0$. For $\lambda \in U$ it follows that $\|\widetilde{x}(\lambda)\| \leqslant \delta^{-1}$. Since $\{\widetilde{x}_n(\cdot)\}$ forms a normal family on U, a subsequence $\widetilde{x}_{n_k}$ converges to an analytic vector valued function f on U (uniformly on compact subsets). For $\lambda_0 \in U$, it follows that

$$(T - \lambda_0)f(\lambda_0) = \lim_{k \to \infty} (T - \lambda_0)\widetilde{x}_{n_k}(\lambda_0) = \lim_{k \to \infty} x_{n_k} = x.$$

By the uniqueness of extension f must be an analytic extension of $(\lambda - T)^{-1}x$ to U. Thus $\sigma_T(x) \subset F$ when $x \in M$.

COROLLARY. *Let $T \in L(H)$ satisfy the conditions in the previous corollary. Then T satisfies condition* C.

THEOREM 3. *Let $T \in L(H)$ be hyponormal. If there exists a nonzero vector $x \in H$ such that $\sigma_T(x) \neq \sigma(T)$ then T has a nontrivial invariant subspace.*

PROOF. Since we can assume without loss of generality that $\sigma(T) = \sigma_C(T)$ the statement of the theorem makes sense. Consider the subspace

$$M = \{y \in H : \sigma_T(y) \subset \sigma_T(x)\}.$$

It is closed, nonempty, invariant under T ($\widetilde{Tx}(\cdot) = T\tilde{x}(\cdot)$) and cannot be all of H since in that case $\sigma(T)$ would equal $\sigma_T(x)$ (see [9, Lemma 1.8]).

COROLLARY. *Let $T \in L(H)$ be hyponormal with $\|T\| = 1$. If there exists a nonzero vector $x \in H$ such that $\|T^n x\| \leqslant Cr^n$ for $n = 1, 2, \ldots$ where $0 < r < 1$ and C is a constant, then T has a nontrivial invariant subspace.*

PROOF. We may assume that $\sigma(T) = \sigma_C(T)$. Let $D_r = \{z : |z| \leqslant r\}$. We claim that $\sigma_T(x) \subset D_r$. Indeed set $\tilde{x}(\lambda) = \Sigma_{n=0}^{\infty} \lambda^{-(n+1)} T^n x$. Clearly $\tilde{x}$ is well defined, analytic for $|\lambda| > r$, and agrees with $(T - \lambda)^{-1}x$ for $|\lambda| > 1$. Since $(T - \lambda)\tilde{x}(\lambda) = x$ for $|\lambda| > r$ we have established the claim. Thus $\sigma_T(x) \neq \sigma(T)$ since the norm of a hyponormal operator is equal to its spectral radius.

It is interesting to note in connection with Theorem 3 that implicitly contained in the work of C. R. Putnam is the fact that if T^* is hyponormal then there exists a nonzero vector $x \in H$ such that $\sigma_T(x)$ is small. More precisely we have

SCHOLIUM (PUTNAM [14]). *Let $T \in L(H)$ be cohyponormal with $\|T\| = 1$ and $\sigma(T) = \sigma_C(T)$. Then there exists a nonzero vector $u \in H$ such that $\sigma_T(u) \neq \sigma(T)$.*

PROOF. Under the condition of the Scholium, Putnam shows that there exists a vector $x \in H$ for which $f_y(z) = ((T - z)^{-1}x, y)$ is continuous on C for each $y \in H$. Then for some $1 > r > 0$ the integral $\int_{|z|=r} (T - z)^{-1}x, x)dz \neq 0$. Set $u = \int_{|z|=r} (T - z)^{-1} x dz$. (More explicitly $(u, y) = \int_{|z|=r} ((T - z)^{-1}x, y)dz$ for each $y \in H$.) We claim that u is the vector we are seeking. Clearly $u \neq 0$ and

$$\tilde{u}(\lambda) = \int_{|z|=r} (z - \lambda)^{-1}(T - z)^{-1} x dz$$

defines an analytic extension of $(T - \lambda)^{-1}u$ for $|\lambda| > r$ as may be easily checked. Thus $\sigma_T(u) \subset D_r$ as desired.

COROLLARY 1. *Let $T \in L(H)$ satisfy the following conditions:*

(1) *there exists a nonzero operator $D \geqslant 0$ such that $(T - z)(T^* - z) \geqslant D$ for all $z \in$ C;*

(2) $\|(T-z)^{-n}x\| \geqslant \|(T-z)^{-1}x\|^n$ *for all* $x \in \operatorname{domain}(T-z)^{-n}$ *and* $n = 1, 2, \ldots$.

Then T has a nontrivial invariant subspace.

PROOF. We may assume $\sigma(T) = \sigma_C(T)$. Thus (2) implies that T has property C by the Corollary to Theorem 2. The proof of Putnam's results and the Scholium really require only (1) to establish the existence of a vector $u \in H$ with $\sigma_T(u) \neq \sigma(T)$. Thus $M = \{y \in H: \sigma_T(y) \subset \sigma_T(u)\}$ is our invariant subspace.

DEFINITION. Let $T \in L(H)$ where $\sigma(T) = \sigma_C(T)$. If there exists a constant K such that $\sigma_T(x) \cap \sigma_T(y) = \varnothing$ implies $\|x\| \leqslant K\|x + y\|$ (independently of x and y) then T satisfies condition B.

COROLLARY 2. *Let $T \in L(H)$ be cohyponormal. If T satisfies either condition* B *or* C *then T has a nontrivial invariant subspace.*

PROOF. We may assume $\sigma(T) = \sigma_C(T)$. Thus we have already covered the C case. To handle B we must modify slightly the construction in the Scholium. Let S be a square which contains $\sigma(T)$. Thus

$$\frac{1}{2\pi i}\int_{\partial S} f_x(\lambda)d\lambda = \frac{1}{2\pi i}\int((T-\lambda)^{-1}x,\ x) = \|x\|^2 \neq 0.$$

Since f_x is continuous we can divide S into two rectangles R_1, R_2 such that $\int_{\partial R_i} f_x(\lambda)dx \neq 0$ for $1 = 1, 2$. Again by continuity, perturb the edge of each rectangle so that we obtain disjoint rectangles J_i where $\int_{\partial J_i} f_x(\lambda)dx \neq 0$. Now define

$$(u_i, y) = \int_{J_i} f_y(\lambda)d\lambda$$

for each $y \in H$ as before. Thus $\sigma_T(u_i) \subset J_i$ as before. Let $M = \{x \in H: \sigma(x) \subset \sigma_T(u_1)\}$. Then M is a nonempty manifold invariant under T but it may not be closed. However for any $x \in M$, $\|x\| \leqslant K\|x + u_2\|$ which implies that $\operatorname{dist}[u_2, M] > 0$. Thus $\operatorname{dist}[u_2, \overline{M}] > 0$. Hence $\overline{M}$ is a nonempty invariant subspace of T which cannot equal H by the last remark.

LEMMA 5. *Let $W, S, T \in L(H)$. Assume that S and T both have the single valued extension property and $TW = WS$. Let $x \in H$; then $\sigma_T(Wx) \subset \sigma_S(x)$.*

PROOF. By hypothesis $\tilde{x}(\cdot)$ is an analytic function on $\rho_s(x)$. Define $f(\lambda) = W\tilde{x}(\lambda)$. Clearly f is analytic on $\rho_s(x)$. Then

$$(T-\lambda)f(\lambda) = (T-\lambda)W\tilde{x}(\lambda) = W(S-\lambda)\tilde{x}(\lambda) = Wx.$$

Thus f is an analytic extension of $(T-\lambda)^{-1}Wx$ and $\sigma_T(Wx) \subset \sigma_S(x)$. (Caution: Wx could be the zero vector.)

THEOREM 4. *Let $S, T, W \in L(H)$ where T is hyponormal, S is cohyponormal and W is injective. Assume that $TW = WS$. Then T has a proper invariant subspace.*

PROOF. First consider the case when S has an eigenvalue. If $Sx = \lambda x$ then $TWx = \lambda Wx$ so we are done. If S^* has an eigenvalue then so does S and again we are finished. So we may assume that neither T nor S has point or residual spectrum. Further we may assume the spectrum of T contains at least two points λ_1 and λ_2. By a slight modification of the technique in the proof of Corollary 2 we can choose an $x \in H$ such that $\sigma_S(x)$ is small; in fact $\operatorname{diam} \sigma_S(x) < |\lambda_1 - \lambda_2|$. Then $Wx \neq 0$ since W is injective and $\sigma_T(Wx) \subset \sigma_S(x)$. Thus $\sigma_T(Wx) \neq \sigma(T)$ and the result now follows from Theorem 3.

REMARK. Relative to the last result we remark that it is an open question as to whether $TW = WS$ with T hyponormal, S normal and W invertible implies T is normal.

We next present another proof of a Theorem of Putnam.

COROLLARY [14]. *Let $T \in L(H)$ be hyponormal where $\|T\| = 1$. If there exists a vector $x_0 \in H$ such that $\|T^{*n}x_0\| \geqslant \alpha > 0$ then T has a proper invariant subspace.*

PROOF. For convenience let $S = T^*$. We may assume that $\ker S^* = \{0\}$. Then $u_n(x) = \|S^n x\|$ is monotone decreasing in n for each x and by a well-known argument (see [18]) there exists a positive operator A such that $(A^2x, x) = \lim_{n\to\infty}(S^*S^n x, x)$ for all $x \in H$. Furthermore A satisfies the equation $S^*A^2S = A^2$; thus $\|ASx\| = \|Ax\|$. It is easy to see that $\ker A$ is an invariant subspace for S so we may assume $\ker A = \{0\}$. (Note that $Ax_0 \neq 0$.) By a theorem of Douglas [8] there exists an isometry C such that $AS = CA$. Thus $S^*A = AC^*$ and since the left side has no kernel, C must be unitary. Thus $TA = AC^*$ where C^* is normal, A is injective and the result follows from the previous theorem.

We will now show that a variation on the hypothesis of the Corollary places a very stringent restriction on the operator if it is subnormal.

THEOREM 5. *Let $T \in L(H)$ be subnormal where $\|T\| = 1$. Assume that $\lim_{n\to\infty} \|T^{*n}x\| \geqslant \epsilon_x > 0$ for all $x \in H$; $x \neq 0$. Then T is unitary.*

PROOF. Again let $S = T^*$. Proceeding along the lines of the previous proof we again obtain a positive operator A. ($(A^2x, x) = \lim_{n\to\infty}(S^n x, S^n x)$ for each $x \in H$.) Of necessity A has trivial kernel. Reasoning as before, $TA = AC^*$ where C is unitary. Since A is injective and intertwines the subnormal operator T and the normal operator C^* (the order is crucial), by a result of Radjavi and Rosenthal [15] or Kulkarni [12], the operators T and C^* are unitarily equivalent, which completes the proof.

§II

We begin the section with an invariant subspace theorem which is different in spirit from those in the previous section although not unrelated. We will then prove a converse which places limitations on the usefulness of Theorem 3.

PROPOSITION 1. *Let $T \in L(H)$ be subnormal with minimal normal extension $B = \int zE(z)$ defined on $K \supset H$. If there exists a nonzero vector $x \in H$ such that $E(\delta)x = x$ for a closed set δ properly contained in $\sigma(T)$, then T has a nontrivial invariant subspace.*

PROOF. We may assume that $\sigma(T) = \sigma(B)$ since all points in $\sigma(T)\backslash\sigma(B)$ are in the residual spectrum of T. We assume T has a cyclic vector in which case T is just multiplication by z on $H^2(1, z, \ldots; d\mu)$ for the appropriate measure $d\mu$ (see [3]).

In this representation $E(\cdot) = \mu(\cdot)$ and $x = f_x \in H^2(1, z, \ldots, d\mu)$. Since $E(\delta)f_x = f_x$; f_x must vanish a.e. μ off δ. Let $M = \mathrm{clm}\{z^n f_x: n = 0, 1, 2, \ldots\}$. Clearly M is invariant under T and since f_x vanishes off δ; $M \neq H$ which completes the proof.

We will now show that the hypothesis in Theorem 3 roughly entails the hypothesis in Proposition 1, at least for subnormal operators. First we will need some intermediary results which are perhaps of independent interest.

LEMMA 6. *Let $T \in L(H)$ be subnormal with minimal normal extension $B \in L(K)$ (thus $K \supset H$). If T^{-1} exists as a densely defined operator and $x \in$* domain *T^{-1} then B^{-1} exists in the same sense, $x \in$* domain *B^{-1} and $T^{-1}x = B^{-1}x$.*

PROOF. If $T^{-1}x = y$, then $By = Ty = x$ and thus $x \in$ domain B^{-1} and $B^{-1}x = y$ provided B^{-1} exists as an unbounded operator. Thus assume to the contrary that $B(f_1 \oplus f_2) = 0$ where $f_1 \in H$ and $f_2 \in H^1$. Then $B^*(f_1 \oplus f_2) = 0$ which requires $PB^*f_1 = 0$ where P is the projection of K on H. But then $T^*f_1 = PB^*f_1 = 0$. Since ker $T^* =$ (range $T)^\perp$, T cannot have an unbounded inverse unless $f_1 = 0$. But then $B(0 \oplus f_2) = B^*(0 \oplus f_2) = 0$ which contradicts the minimality of B.

The next lemma should appear in the Dunford work on spectral operators but we know of no reference.

LEMMA 7. *Let $B \in L(H)$ be normal where $B = \int zdE(z)$. Let $x \in H$. Then $E(\rho_B(x))x = 0$.*

PROOF. Since B is normal it has the single valued extension property and $\sigma_B(x)$ is well defined. It follows from Lemma 1 that if $\lambda \in \rho_B(x)$ then $x \in \mathrm{dom}(B - \lambda)^{-1}$. (Actually, Lemma 1 contains the added hypothesis that $\lambda \in \sigma_C(B)$ but it is easy to drop this condition when the operator is normal.) Set

$$Q = \{\lambda \in C: x \in \mathrm{dom}(B - \lambda)^{-1}\}.$$

Then by a result of Dixmier and Foiaş [7] Q is a Borel set, indeed a F_σ. (This result does not require normality.) Clearly $Q \supset \rho_B(x)$. But Putnam [13], has shown that $E(Q)x = 0$ which completes the proof.

I am grateful to Medhi Radjabalipour for a suggestion which considerably simplifies the proof of the next theorem.

THEOREM 6. *Let $T \in L(H)$ be subnormal with minimal normal extension $B = \int z dE(z)$. Let $x \in H$. Then $E(\rho_T(x)) = 0$. If T has a cyclic vector so that T is multiplication by z on $H^2(1, z, \ldots, d\mu)$ and x corresponds to f_x then $f_x = 0$ a.e. μ on $\rho_T(x)$.*

PROOF. By hypothesis there exists an analytic vector valued function $\tilde{x}(\cdot)$ such that $(T - \lambda)\tilde{x}(\lambda) = x$ for all $\lambda \in \rho_T(x)$. Thus $(B - \lambda)\tilde{x}(\lambda) = x$ for $\lambda \in \rho_T(x)$ which implies that $\rho_B(x) \supset \rho_T(x)$. Since $E(\rho_B(x)x) = 0$ by the previous lemma this completes the first part of the proof. The final assertion is merely a restatement of the foregoing material.

§III

This section addresses itself to the question: Must a hyponormal quasi-triangular operator be normal? What if it is very quasi-triangular?

DEFINITION. An operator $T \in L(H)$ is quasi-triangular if there exists a sequence $\{E_n\}$ of finite dimensional selfadjoint projections such that $E_n \leqslant E_{n+1}$ for $n = 1, 2, \ldots, E_n \longrightarrow I$ strongly and finally $\|(1 - E_n)TE_n\| \longrightarrow 0$.

We begin by presenting two examples of hyponormal quasi-triangular operators which are not normal.

EXAMPLE 1. Let S be the unilateral shift and let M_z be the multiplication operator on $L^2(\Delta, dm)$ where Δ is the unit disc in the complex plane C and dm is area measure on Δ (M_z: $f(z) \longrightarrow zf(z)$). Then $S \oplus M_z$ has the form "normal + compact" by [6] and hence is quasi-triangular by [11]. Clearly $S \oplus M_z$ is hyponormal; in fact, it is the direct sum of a subnormal and a normal operator.

EXAMPLE 2. Clancey and Morrell [5] have observed that an example of J. Brennan's [4] produces a subnormal operator T with no residual spectrum. By the remarkable result of Apostol, Foiaş and Voiculescu [1], T must be quasi-triangular.

LEMMA 8. *Let $T \in L(H_n)$ where H_n is n-dimensional. Assume $\|T^*f\|^2 \leqslant \|Tf\|^2 + \delta^2\|f\|^2$ for all $f \in H_n$. Then $\|Tf\|^2 \leqslant \|T^*f\|^2 + 4\|T\|n\delta\|f\|^2$.*

PROOF. Write T in upper triangular form with respect to the basis $\varphi_1, \ldots, \varphi_n$ so that $T = [a_{ij}]$ in this basis.

We then obtain the relation

$$\|T^*\varphi_i\|^2 = \sum_{j=1}^{n} |a_{i,j}|^2 \leqslant \delta^2 + \sum_{k=1}^{n} |a_{k,i}|^2 = \|T\varphi_i\|^2 + \delta^2.$$

Hence for $i > k$ we have

$$|a_{k,i}|^2 \leqslant \delta^2 + \sum_{l=1}^{k-1} |a_{l,k}|^2 - \sum_{p=k+1;p\neq i}^{n} |a_{k,p}|^2.$$

Thus

$$\sum_{j=i+1}^{n} |a_{i,j}|^2 \leqslant \delta^2 + \sum_{k=1}^{i-1} \left[\delta^2 + \sum_{l=1}^{k-1} |a_{l,k}|^2 - \sum_{p=k+1;p\neq i}^{n} |a_{k,p}|^2\right].$$

But

$$\sum_{k=1}^{i-1} \sum_{l=1}^{k-1} |a_{l,k}|^2 - \sum_{k=1}^{i-1} \sum_{p=k+1;p\neq i}^{n} |a_{k,p}|^2$$

$$= \sum_{k=1;k>l}^{i-1} \sum_{l=1}^{k-1} |a_{l,k}|^2 - \sum_{k=l+1;k\neq i}^{n} \sum_{l=1}^{i-1} |a_{l,k}|^2$$

and this last expression is clearly negative since $a_{l,k} = 0$ for $k < l$.

Thus $\Sigma_{j=i+1}^{n} |a_{i,j}|^2 \leqslant i\delta^2$. If we write $T = D + R$ where D is the diagonal part of T, then

$$\|R\|_{\text{H.S.}}^2 = \sum_{i=1}^{n} \sum_{j=i+1}^{n} |a_{i,j}|^2 \leqslant \sum_{i=1}^{n} i\delta^2 = \frac{n(n+1)}{2}\delta^2.$$

Thus $\|R\|_{\text{H.S.}} \leqslant n\delta$ where $\|\cdot\|_{\text{H.S.}}$ denotes the Hilbert-Schmidt norm of R. For any $f \in H_n$ with $\|f\| = 1$ it follows that

$$\|Tf\| \leqslant \|Df\| + \|Rf\| \quad \text{and} \quad \|T^*f\| \geqslant \|D^*f\| - \|R^*f\|$$

whenever $\|Tf\| - \|T^*f\| \leqslant 2n\delta$. Multiplying by $\|Tf\| + \|T^*f\|$ we find

$$\|Tf\|^2 \leqslant \|T^*f\|^2 + 4\|T\|n\delta \quad \text{for all } f \in H_n$$

where $\|f\| = 1$, which completes the proof.

COROLLARY. *Let $T \in L(H_n)$ where $\|T^*\|^2 \leqslant \|Tf\|^2 + \delta^2\|f\|^2$ for all $f \in H_n$. Then $T = D + R$ where D is diagonal and R is a nilpotent operator whose Hilbert-Schmidt norm is less than $n\delta$.*

For those who would prefer a version without the quadratic terms we have

COROLLARY. *Let $T \in L(H_n)$ where $\|T^*f\| \leqslant \|Tf\| + \delta\|f\|$ for all $f \in H_n$ with $\|f\| = 1$. Then $\|Tf\| \leqslant \|T^*f\| + 2n\sqrt{2\delta\|T\|}$.*

THEOREM 7. *Let $T \in L(H_n)$ be a hyponormal operator. Assume $\|TE_n - E_nTE_n\| = \theta_n$ where the E_n's form an increasing sequence of projections converging* *to I and each E_n has rank n. If* $\lim\inf_{n\to\infty} n\theta_n = 0$; *then T is normal.*

PROOF. Assume $f \in E_m H$ for some fixed m. Then for $n > m$ we write

$$T = \begin{vmatrix} A & C \\ B & D \end{vmatrix} \quad \text{on } E_n H \oplus E_n H^{\perp}.$$

Since T is hyponormal (and $B = TE_n - E_n TE_n$)

$$\|Tf\|^2 = \|Af\|^2 + \|Bf\|^2 \geqslant \|A^*f\|^2 + \|C^*f\|^2 = \|T^*f\|^2$$

and hence $\|Af\|^2 + \theta_n^2 \geqslant \|A^*f\|^2$. Applying the lemma to the operator A we find that

$$\|Tf\|^2 = \|Af\|^2 + \|Bf\|^2 \leqslant \|A^*f\|^2 + 4\|A\| n\theta_n + \theta_n^2$$

$$\leqslant \|T^*f\|^2 + 4\|A\| n\theta_n + \theta_n^2.$$

But with appropriate choice of n this implies that $\|Tf\| \leqslant \|T^*f\|$ whence $\|Tf\| = \|T^*f\|$. Since the set of f's for which equality holds is dense in H one must conclude that T is normal.

COROLLARY. *Let* $T \in L(H)$ *be hyponormal. If there exists an* $f \in H$, $\|f\| = 1$, *such that* $\lim\inf_{n\to\infty} n\|T^n f\|^{1/n} \longrightarrow 0$ *then T has a proper invariant subspace.*

PROOF. Consider the sequence $f, Tf, T^2f, \ldots$ which must be dense in H else we are done. Apply the Gram-Schmidt process to this sequence to obtain an orthonormal basis $e_1, e_2, \ldots$ for H. By construction $Te_n = a_n e_{n+1} + g_n$ where $g_n \perp e_j$ for $j \geqslant n+1$, $n = 1, 2, \ldots$. If we let E_n be the projection of H on $\operatorname{clm}\{e_1, \ldots, e_n\}$ then $\|(1 - E_n)TE_n\| = |a_n|$. But $T^n f = T^n e_1 = a_1 \cdots a_n e_{n+1} + h$ where $h \perp e_j$ for $j \geqslant n+1$. Thus $n\|T^n f\|^{1/n} \geqslant n|a_1 \cdots a_n|^{1/n}$. Hence $\lim\inf_{n\to\infty} n|a_n| \longrightarrow 0$ which implies that T is normal by the previous theorem so we are done.

We will now present an example which shows that, while the estimate obtained in the Lemma 8 may not be best possible, it is not too far off.

EXAMPLE. Let $\{\varphi_1, \ldots, \varphi_n\}$ be an orthonormal basis for H. Define

$$T\varphi_j = \begin{cases} 0, & j = 1, \\ \sqrt{j/n}\,\varphi_{j-1}, & \text{for } j = 2, \ldots, n. \end{cases}$$

Then $\|T^*f\|^2 \leqslant \|Tf\|^2 + n^{-1}\|f\|^2$ for all $f \in H_n$ and thus we may take $\delta = 1/\sqrt{n}$. But $\|T\varphi_n\|^2 \simeq \|T^*\varphi_n\|^2 + 1$. Thus $\|T\varphi\|^2 \leqslant \|T^*\varphi_n\|^2 + 2\|T\|\sqrt{n\delta}$ in this example. However $\sqrt{n}$ *cannot* be replaced by n^a for any $a < 1/2$.

ADDED IN PROOF. By an ingenious modification of Lemma 3, M. Radjabalipour has shown that the lemma and consequently Theorems 1 and 2 are valid for any hyponormal operator. (In other words, the condition $\sigma_R(T) = \varnothing$ is superfluous.) The question raised in the Remark following Theorem 4 has been answered (independently and affirmatively) by Fong and Radjabalipour, and by Stampfli and Wadhwa. Finally Theorem 5 has been extended by C. R. Putnam to cover hyponormal operators and by the present author to cover dominant operators.

REFERENCES

1. C. Apostol, C. Foiaş and D. Voiculescu, *Some results on non-quasitriangular operators.* IV, Rev. Roumaine. Math. Pures Appl. **18** (1973), 487–514. MR **48** #12109a.

2. R. G. Bartle, *Spectral localization of operators in Banach spaces,* Math Ann. **153** (1964), 261–269. MR **29** #1542.

3. J. Bram, *Subnormal operators,* Duke Math. J. **22** (1955), 75–94. MR **16**, 835.

4. J. E. Brennan, *Invariant subspaces and rational approximation,* J. Functional Analysis **7** (1971), 285–310.

5. K. Clancey and B. Morrell, *The essential spectrum of some Toeplitz operators* (to appear).

6. J. A. Deddens and J. G. Stampfli, *On a question of Douglas and Fillmore,* Bull. Amer. Math. Soc. **79** (1973), 327–330. MR **47** #7483.

7. J. Dixmier and C. Foiaş, *Sur le spectre ponctuel d'un opérateur,* Colloq. Math. Soc. János Bolyai (5) Hilbert Space Operators, Tihany, Hungary, 1970, pp. 127–133.

8. R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space,* Proc. Amer. Math. Soc. **17** (1966), 413–415. MR **34** #3315.

9. N. Dunford, *Spectral theory.* II. *Resolutions of the identity,* Pacific J. Math. **2** (1952), 559–614. MR **14**, 479.

10. N. Dunford and J. T. Schwartz, *Linear operators.* III, *Spectral operators,* Interscience, New York, 1971.

11. P. R. Halmos, *Quasitriangular operators,* Acta. Sci. Math. (Szeged) **29** (1968), 283–293. MR **38** #2627.

12. R. Kulkarni, *Subnormal operators and weighted shifts,* Thesis, Indiana University, 1970.

13. C. R. Putnam, *Ranges of normal and subnormal operators,* Michigan Math. J. **18** (1971), 33–36. MR **43** #2550.

14. ———, *Resolvent vectors, invariant subspaces and sets of zero capacity,* Math. Ann. **205** (1973), 165–171. MR **48** #4772.

15. H. Radjavi and P. Rosenthal, *On roots of normal operators,* J. Math. Anal. Appl. **34** (1971), 653–664. MR **43** #3829.

16. W. Rudin, *Functional analysis,* McGraw-Hill, New York, 1973.

17. J. G. Stampfli, *Analytic extensions and spectral localization,* J. Math. Mech. **16** (1966), 287–296. MR **33** #4687.

18. B. Sz. Nagy, *On uniformly bounded linear transformations in Hilbert space,* Acta Univ. Szeged Sect. Sci. Math. **11** (1947), 152–157. MR **9**, 191.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125